1. Review

Recall our definitions regarding probability. We perform some sort of "experiment", such as rolling a die. An *outcome* is result of the experiment; in this case, the outcome is an integer between one and six. The *sample space* is the set of all possible outcomes; in this example, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$. We may be looking for certain outcomes, say for example, we wish to roll and even number. An *event* is a subset of the sample space, say $E = \{2, 4, 6\}$.

The *cardinality* of a set is the number of things in it. The cardinality of the set A is denoted $|A|$. In our example, $|S| = 6$ and $|E| = 3$.

The *probability* of event E is defined to be

$$P(E) = \frac{|E|}{|S|}.$$

2. Motivational Exmaple

It quickly becomes apparent that computing probabilities, according to the definition, is a problem of counting members of a set.

Example 1. Suppose we have five balls, numbered one through five. We put the balls in a basket, and randomly draw out one at a time, until all five are withdrawn. What is the probability that the balls were selected in increasing order?

Discussion. An outcome here is a sequence of five distinct integers, in order. We write this as an ordered quintuple, such as $(2, 3, 5, 1, 4)$.

Let $B = \{1, 2, 3, 4, 5\}$. This represents the set of balls. The sample space S is the set of ordered quintuples from the set B with distinct entries.

The event is $E = \{(1, 2, 3, 4, 5)\}$. So, the probability is

$$P(E) = \frac{|E|}{|S|} = \frac{1}{|S|}.$$

But what is $|S|$? It is the number of ways of arranging five things. □

3. Factorials

Suppose we have a set of n objects. We wish to count the number of ways there are to arrange (or rearrange) these objects in a list. We image a sequence of n slots, and realize that we wish to place one of the objects in each slot. We have n choices for the first slot, $n-1$ choices for the second slot, so the number of ways to begin our arrangement with its first two elements is $n(n-1)$, by the multiplication principle. Continuing, we have $n-2$ ways to choose the third object, and so forth, continuing to multiply until we see that there are

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

ways to arrange n objects.

If n is a positive integer, mathematicians define n *factorial*, written $n!$, to be the product of the positive integers less than or equal to n. That is,

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n.$$

So, the number of ways of arranging n things is $n!$. For convenience, we also define $0!$ to be 1. The first few factorials are:

$$\begin{aligned}
0! &= 1\\
1! &= 1\\
2! &= 2\\
3! &= 6\\
4! &= 24\\
5! &= 120\\
6! &= 720\\
7! &= 5040\\
8! &= 40320\\
9! &= 382880
\end{aligned}$$

That is, there are $n!$ different lists of distinct objects we can make out of a set of n objects.

We now solve our motivational problem.

Example 1 (continued)**.** Suppose we have five balls, numbered one through five. We put the balls in a basket, and randomly draw out one at a time, until all five are withdrawn. What is the probability that the balls were selected in increasing order?

Solution. Let $B = \{1, 2, 3, 4, 5\}$. This represents the set of balls. The sample space S is the set of ordered quintuples from the set B with distinct entries.

Now S may be viewed as the set of all ways of arranging distinct members from the set B. Let $n = |B| = 5$; then $|S| = n! = 5! = 120$.

The event is $E = \{(1, 2, 3, 4, 5)\}$, which has one thing in it. So, the probability is

$$P(E) = \frac{|E|}{|S|} = \frac{1}{|S|} = \frac{1}{120} = 0.008333.$$

□